Notes 3-6: Critical Points and Extrema
I. Critical Points
A. Definition and Types of Critical Points
• Critical Points: those points on a graph at which a line drawn tangent to the curve is horizontal or vertical.
• Polynomial equations have three types of critical points—maximums, minimums, and points of inflection.
B. Relative vs. Absolute

• The term ‘extrema’ refers to maximums and/or minimums.
• The general term for maximums or minimums is ‘extremum’.
• Extrema can be relative or absolute.
• An absolute minimum/maximum is the greatest/least value that a function assumes over its domain.
• A relative max/min may not be the greatest/least over its domain, but it is the greatest/least over some interval in the domain.
• Extrema are always values of the function; they are the y-coordinates of each max or min.
C. Finding Extrema Given Graph

Locate the extrema for the following graphs. Name and classify the extrema of the function. Remember, extrema is just max and min, NOT points of inflection.

Ex. 1:

The function has an absolute minimum at (-6, -6). The absolute minimum is -6.

The function has a relative minimum at (1, -2) The relative minimum is -2.

The function has a relative maximum at (-2, 1). The relative maximum is 1.

There is no absolute maximum, because the graph goes up forever and ever... and ever.
Ex. 2: Locate the extrema for the following graphs. Name and classify the extrema of the function.

Relative max at (0, 2). The relative maximum is 2.

Relative min at (2, -2). Relative min s -2.

No absolute max or min.
II. Finding Critical Points with Calculator.

• Three methods:
  – Use a table of values
  – Use the trace function
  – Use the 3: minimum and 4: maximum options in the CALC menu
Ex 1: Use a GC to graph \( f(x) = 5x^3 - 10x^2 - 20x + 7 \) and to determine and classify its extrema.

- Graph using standard viewing window.
- Notice where the x-intercepts are - relative extrema will be in between these points.
- Adjust the viewing window to get a better view of the graph.
Now, find the extrema.

• Method 1: Use a table of values. Since it looks like a relative max is somewhere around -1, set your table to start there and increase at increments of 0.1. Do the same thing at 2.

There seems to be a relative max of app. 14.385 at x=-0.7, and a relative min of -33 at 2.
• Method 2: You can also trace the function to locate the approximate extrema. Try it.

Method 3: CALC>> 3: minimum or 4: maximum.
Note: You will be asked to define a left and right bound. Choose a point to the left, and then right of the max/min. Then it will “guess” the max/min.

All three methods are approximations.
III. Given Critical Point, Classify It

Notice for a maximum, y values on both the left and right of the maximum are smaller than the y-value at the maximum. For example, if the max occurs where \( x = 2 \), \( f(1.9) \) and \( f(2.1) \) are smaller than \( f(2) \).

Notice for a minimum y values on both the left and right of the minimum are larger than the y-value at the minimum. For example, if the min occurs where \( x = 2 \), \( f(1.9) \) and \( f(2.1) \) is larger than \( f(2) \).

For a point of inflection, the y-value on the left is smaller, the y-value on the right is larger (or vice versa). Therefore, if the point of inflection is at \( x=2 \), then either \( f(1.9) \) is smaller than \( f(2) \) and \( f(2.1) \) is larger than \( f(2) \). Or \( f(1.9) \) is larger than \( f(2) \) and \( f(2.1) \) is smaller than \( f(2) \).
### Critical Points

For $f(x)$ with $(a, f(a))$ as a critical point and $h$ as a small value greater than zero

<table>
<thead>
<tr>
<th>Condition</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(a - h) &lt; f(a)$, $f(a + h) &lt; f(a)$</td>
<td>$f(a)$ is a maximum.</td>
</tr>
<tr>
<td>$f(a - h) &gt; f(a)$, $f(a + h) &gt; f(a)$</td>
<td>$f(a)$ is a minimum.</td>
</tr>
<tr>
<td>$f(a - h) &gt; f(a)$, $f(a + h) &lt; f(a)$</td>
<td>$f(a)$ is a point of inflection.</td>
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<td>$f(a - h) &lt; f(a)$, $f(a + h) &gt; f(a)$</td>
<td>$f(a)$ is a point of inflection.</td>
</tr>
</tbody>
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Ex 1: The function $f(x) = 2x^5 - 5x^4 - 10x^3$ has critical points at $x = -1$, $x = 0$, and $x = 3$. Determine whether each of these critical points is the location of a maximum, a minimum, or a point of inflection.

For each value, test an $x$-value slightly smaller and slightly larger than that $x$-value.

- If both are smaller than $f(x)$, then it is a maximum.
- If both are larger than $f(x)$, then it is a minimum.
- If one is smaller and the other is larger than $f(x)$, then it is an inflection point.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f$(Smaller $x$)</th>
<th>$f(x)$</th>
<th>$f$(Larger $x$)</th>
<th>Type of Critical Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>$f(-1.1) = 2.769$ smaller</td>
<td>$f(-1) = 3$</td>
<td>$f(-0.9) = 2.829$ smaller</td>
<td>maximum</td>
</tr>
<tr>
<td>0</td>
<td>$f(-0.1) = 0.009$ larger</td>
<td>$f(0) = 0$</td>
<td>$f(0.1) = -0.010$ smaller</td>
<td>inflection point</td>
</tr>
<tr>
<td>3</td>
<td>$f(2.9) = -187.308$ larger</td>
<td>$f(3) = -189$</td>
<td>$f(3.1) = -187.087$ larger</td>
<td>minimum</td>
</tr>
</tbody>
</table>
Ex 2: The function $f(x) = 3x^4 - 4x^3$ has critical points at $x = 0$ and $x = 1$. Determine whether each of these critical points is the location of a maximum, minimum, or point of inflection.

For each value, test an $x$-value slightly smaller and slightly larger than that $x$-value.

If both are smaller than $f(x)$, then it is a maximum.
If both are larger than $f(x)$, then it is a minimum.
If one is smaller and the other is larger than $f(x)$, then it is an inflection point.

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<tr>
<th>$x$</th>
<th>$f($Smaller $x)$</th>
<th>$f(x)$</th>
<th>$f($Larger $x)$</th>
<th>Type of Critical Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$f(0.9) = -.9477$</td>
<td>$f(1) = -1$</td>
<td>$f(1.1) = -.9317$</td>
<td>minimum</td>
</tr>
<tr>
<td>0</td>
<td>$f(-0.1) = 0.0043$</td>
<td>$f(0) = 0$</td>
<td>$f(0.1) = -0.0037$</td>
<td>inflection point</td>
</tr>
</tbody>
</table>
IV. Applications

• One hour after $x$ milligrams of a particular drug are given to a person, the rise in body temperature $T(x)$, in degrees Fahrenheit, is given by $T(x) = x - \frac{x^2}{9}$. The model has a critical point at $x = 4.5$.

a. Determine if this is a maximum.

b. Why might this be important to a doctor or pharmacist?
One hour after \( x \) milligrams of a particular drug are given to a person, the rise in body temperature \( T(x) \), in degrees Fahrenheit, is given by \( T(x) = x - \frac{x^2}{9} \). The model has a critical point at \( x = 4.5 \).

a. Determine if this is a maximum. Use any method.

- Make a table
- Graphing Calculator
  - Look at graph
  - Trace
  - Table

This is a maximum. The greatest temperature change is caused by 4.5 mg of the drug.

b. Why might this be important to a doctor or pharmacist?

Patients should be warned that their body temperatures could fluctuate by 2.25 degrees F.